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Original Research

Comparison of Analytical and Numerical Solutions for Steady, Gradually Varied Open-Channel Flow

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Abstract

The depths for a steady, gradually varied flow in a rectangular open channel with infinite width obtained analytically by the Bresse formula and numerically using trapezoidal rule for integration have been compared. It has been shown that numerical integration may generate significant errors of flow values for looped river networks. The method for choosing the optimum integration step that minimizes numerical integration errors has been presented.

Keywords: open-channel flow, river looped networks, gradually varied flow, Bresse formula, numerical integration

Introduction

Among various open-channel systems, a special position is occupied by looped networks, i.e. networks with cyclic sequences of reaches. Looped structure often happens at irrigation systems; in the case of rivers this this results from natural bifurcations, in particular at deltaic mouths, like the Red River in Vietnam, the Ganges and the Mekong [1] as from hydrotechnical works (bypasses). Rivers of braided and anastomosing types [2] can be perceived as looped networks as well. The Lower Oder River is an example of such a system in Poland [3].

Looped networks are worthy of special regard among others due to the fact that any local change of flow conditions, e.g. the narrowing or deepening of a river bed in one place, may affect the flow values at the major part of the network. Hence, these networks require computational methods that allow us to determine flows and water levels with high accuracy.

The present paper aims to show circumstances of significant errors at flow determination in looped networks using standard numerical methods. Simultaneously, some possibilities of computational accuracy improvement by relevant numerical integration are shown.

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Gradually Varied Flow in Open Channels – Analytical Solutions

The energy equation is the base for all methods of steady, gradually varied flow computations in open channels. Assuming abscissa x directed opposite the flow (Fig. 1), this equation can be presented in a differential form:

$$z + \frac{\alpha v^2}{2g} = z + dz + \frac{\alpha}{2g}\left(v + dv\right)^2 - Sdx \qquad (1)$$

...where z – water surface elevation [m], α – Coriolis coefficient [-], v – mean velocity in a cross-section [m·s^{-1}], g – gravity [m·s^{-2}], S – energy line slope (hydraulic slope) [-], or in an integral one:

$$z_2 - z_1 + \frac{\alpha}{2g}\left(v_2^{\,2} - v_1^{\,2}\right) = \int_{x1}^{x2} Sdx \qquad (2)$$

...where z_1, z_2 are elevations and v_1, v_2 are mean velocities in cross-sections positioned at x_1 and x_2, respectively.

Considering the rectangular bed cross-section, Eq. (1) can be written as follows:

$$\frac{dh}{dx}=\frac{S-S_0}{1-\left(\frac{h_c}{h}\right)^3} \tag{3}$$

...where h is depth [m], h_c – critical depth [m], S_0 – bed slope [-].

Solution of Eq. (3) requires specification for the energy loss function. Assuming the concept of the infinitely wide channel with a constant bed slope as a base for further considerations, at constant value of Chézy coefficient Eq. (3) is given by:

$$\frac{dh}{dx}=-S_0\frac{1-\left(\frac{h_n}{h}\right)^3}{1-\left(\frac{h_c}{h}\right)^3} \tag{4}$$

...where h_n is the normal depth [m], while Manning's approach leads to the equation:

$$\frac{dh}{dx}=-S_0\frac{1-\left(\frac{h_n}{h}\right)^{10/3}}{1-\left(\frac{h_c}{h}\right)^3} \tag{5}$$

The analytical solution of Eq. (4) is known as the Bresse formula [4], which is for the given initial condition (x_0, h_0) and at the following denotations:

$$\frac{h}{h_n}=y,\quad \frac{h_c}{h_n}=\beta,\quad \beta^3=Fr^2 \tag{6}$$

...where Fr is a Froude number for uniform flow and may be written as follows:

$$\frac{S_0(x-x_0)}{h_n}=F(y_0)-F(y) \tag{7}$$

...where:

$$F(y)=y+\frac{1}{6}\left(1-Fr^2\right)\cdot\left[\ln\frac{(y-1)^3}{y^3-1}+2\sqrt{3}\cdot\arctan\frac{\sqrt{3}}{2y+1}\right] \tag{8}$$

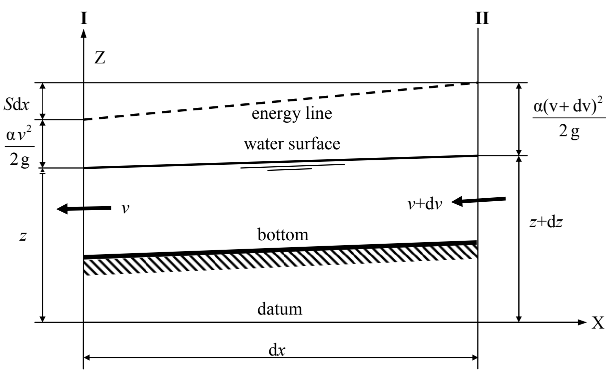

Fig. 1. Gradually varied open-channel flow.

Analytical solution for equation (5) was given by Venutelli [5].

Gradually Varied Flow in Open Channels – Comparison of Numerical and Analytical Solutions

The integral form (2) of a gradually varied flow equation is basically applied in natural channels, while energy losses being an integral of the hydraulic slope function are calculated using the trapezoidal rule method [6]:

$$\int_x^{x+\Delta x} S dx \cong \frac{S(x+\Delta x)+S(x)}{2}\cdot\Delta x \tag{9}$$

This method is widely accepted as a sufficiently accurate one [4, 7], so the applications of other methods of numerical integration, like Runge-Kutta [4, 8] or Kutta-Merson [4], are rare.

Some researchers [1, 9], while discussing methods of flow calculations in looped networks, assume arbitrarily the acceptable error of water elevations in a single integration step as 0.001 m, which creates an impression of accuracy sufficient for practical purposes. Adlul Islam et al. [10], comparing the effectiveness of various algorithms for river network calculations, assume (also arbitrarily) the acceptable error ten times smaller, i.e. 0.0001 m. However, considering the looped network as the trapezoidal method as the mentioned error, values may appear insufficiently accurate, as in the example discussed below.

Example: Let us consider a river network (Fig. 2) consisting of two main channels, 2-1 and 3-1, with constant bed slope joined in node No. 1 and connected additionally by transverse channel 2-3. The following assumptions are made:

- cross-sections for each channel are rectangular with infinite width,
- the Chézy coefficient for each channel is a constant independent of the depth and calculated as for uniform flow due to the Manning formula for the given roughness coefficient n.

These assumptions make the Bresse formula (7, 8) the exact solution for the problem of water surface level determination for each reach of the network. Although Manning formula is basically redundant here due to the invariability of the Chézy coefficient, it was used to demonstrate the proximity of assumptions to a possible real situation.

Using independently the Bresse formula and the integral form of Eq. (2) at different integration steps Δx by

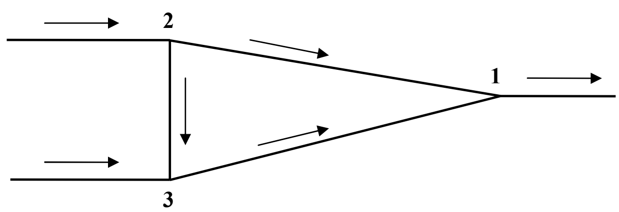

Fig. 2. Example of cyclic looped river network.

Table 1. Determination of flows in an example of a cyclic looped river network.

Assumptions				
Value	**Reach**			
	2-1		**3-1**	
Lengths [m]	2,000		2,000	
Bed slopes [-]	0.001		0.001	
Unit flows [m^2s^{-1}]	3.00		5.00	
Roughness coefficients [$m^{-1/3}s$]	0.032		0.020	
Depth in node No. 1 [m]	3.000			
	Reach 2-3			
Length [m]	200			
Bed slope [-]	0			
Roughness coefficient [$m^{-1/3}s$]	0.026			
Calculation results				
Values common for both methods	**Reach**			
	2-1		**3-1**	
Normal depths [m]	1.95		2.00	
Critical depths [m]	1.00		1.41	
Chézy coefficients [$m^{1/2}s^{-1}$]	34.92		56.10	
Values for particular methods	**Bresse (exact)**	**Δx=500 m**	**Δx=1,000 m**	**Δx=2,000 m**
Depths in node No. 2 [m]	2.05405	2.05277	2.04994	2.05754
Errors of depth in node No. 2 in relation to exact value [mm]	0	-1.28	-4.11	+3.49
Depths in node No. 3 [m]	2.05337	2.05028	2.04189	2.04430
Errors of depth in node No. 3 in relation to exact value [mm]	0	-3.09	-11.48	-9.07
Unit flow at reach 2-3 [m^2s^{-1}]	0.235	0.450	0.805	1.036

trapezoidal rule (9), unit flow at reach 2-3 should be calculated. The following values have been assumed for reaches 2-1 and 3-1:

- unit flows q [m^2s^{-1}],
- Manning roughness coefficient n [$m^{-1/3}s$].

Additionally the following values are given:

- depth in the node No. 1 common for both main reaches h_0 [m],
- constant Coriolis coefficient α=1.1.

The sequence of calculations is presented below:

1. Determination of critical depth, normal depth, and Chézy coefficients for reaches 2-1 and 3-1 due to the relations:

$$h_n = \left(\frac{q \cdot n}{\sqrt{S_0}} \right)^{3/5}, \quad h_c = \sqrt[3]{\frac{\alpha \cdot q^2}{g}}, \qquad C = \frac{1}{n} h_n^{\ 1/6} \tag{10}$$

2. Calculation of depths in nodes 2 and 3 using adopted methods.
3. Determination of unit flow at reach 2-3 using the Chézy-Manning formula.

All assumptions and results of computations performed due to the presented algorithm are shown in Table 1.

Analysis of the calculation results shows that differences between exact water level elevations and the elevations obtained by integration of energy losses using trapezoidal rule method are seemingly small and negligible (a few millimeters); however, these differences are the source of high changes of calculated unit flow values at reach 2-3. This may produce biased results of mathematical modeling of flows in a complicated river network with looped structure and give even entirely false flow conditions in such a network as a consequence. Additionally, the elevation differences are not a monotone function of the integration step length Δx.

Therefore, the problem of accuracy for water level determination is essential when gradually varied flow has been considered by an integral form of the energy equation (2). In particular, the proper choice of the integration steps minimizing the errors of channel depths and elevations determination becomes important. Earlier, this problem was reported by Tavener [11]. Next, Družeta et al. [12], while analyzing the influence of finite element size on the accuracy of the solution for the 2-D open-flow problem, showed the existence of an optimal element size that, when decreased, may produce the worse quality of the model.

One of the possible analysis options is the application of the modified trapezoidal rule due to the formula:

$$\int_{x_1}^{x+\Delta x} S dx = [\lambda \cdot S(x+\Delta x) + (1-\lambda)\cdot S(x)]\cdot \Delta x \tag{11}$$

...where λ is the weight coefficient, and determination of the λ value that results in the numerical integral of the energy head losses equal to the relevant analytical value.

For rectangular channel Eq. (2) yields:

$$h_1 + \frac{\alpha q^2}{2gh_1^2} = h_2 + S_0 \cdot \Delta x + \frac{\alpha q^2}{2gh_2^2} - \int_{x1}^{x1+\Delta x} S dx \tag{12}$$

...where $\Delta x = x_2 - x_1$. Assuming conditions adequate to the Bresse solution the hydraulic slope can be expressed as:

$$S(x) = \frac{q^2}{C^2 h^3} = S_0 \left(\frac{h_n}{h}\right)^3 \tag{13}$$

Therefore, using denotations (6), relations (10), and formula (11), Eq. (12) can be written as:

$$y_1\left[1+\frac{1}{2}\left(\frac{\beta}{y_1}\right)^3\right] = y_2\left[1+\frac{1}{2}\left(\frac{\beta}{y_2}\right)^3\right] - \frac{S_0\cdot\Delta x}{h_n}\left[\lambda\left(\frac{1}{y_2^3}-\frac{1}{y_1^3}\right)+\frac{1}{y_1^3}-1\right] \tag{14}$$

Rewriting Eq. (7) in a form:

$$\frac{S_0 \cdot \Delta x}{h_n} = F(y_1) - F(y_2) \tag{15}$$

...the set of equations is obtained that allows us to determine relevant λ value for any integration step as a function:

$$\lambda = \lambda(\beta, \gamma, y_2) \tag{16}$$

...where:

$$\gamma = \frac{S_0 \cdot \Delta x}{h_n} \tag{17}$$

For practical reasons it is more convenient to show the relation (16) in a logarithmic scale, i.e.:

$$\lambda = \lambda(\beta, \gamma, y^*) \tag{18}$$

...where:

$$y^* = -\log_{10}(y_2 - 1) \tag{19}$$

Calculations of the λ values were performed for the following ranges: $\beta \in \langle 0,1 \rangle$, $\gamma \in \langle 0;1.5 \rangle$, and $y^* \in \langle -1;2 \rangle$. This variability covers practically the whole range of flow parameters at backwater computations. The graphs for some β values are shown in Fig. 3. It results that for the given values of β and γ there exists only one value $y^*=y_0^*$, where λ=0.5 and numerical integration by trapezoidal rule gives an unbiased result. Analogically, at given values β and y^* there exists the single (at the outmost) value $\gamma=\gamma_0$ fulfilling this condition. This means that while performing one integration step for the gradual flow equation (4) by trapezoidal rule, the depth is equal to the exact analytical value at one, if any, length of the spatial integration step $\Delta x = \Delta x_0$. The graphs show that at sufficiently small values y^* (one may estimate $y^*<0$, so at depths $h \geq 2 \cdot h_n$) the value $\Delta x_0 > 0$ does not exist; therefore, every integration step must be biased. Worthy of notice is that at $y^* \cong y_0^*$, plots $\lambda=\lambda(\beta,\gamma,y^*)$ have point of inflexion and high gradients that cause each inexactness of y_0^* determination affecting λ value relatively strongly.

Substituting λ=0.5 and putting $\gamma=\gamma_0$ to Eq. (18) the following inverse function can be found:

$$y_0^* = y_0^*(\beta, \gamma_0) \tag{20}$$

The graphs for relation (20) are shown at Fig. 4, whilst coefficients of the approximating trinomial function

$$y_0^* = a_0(\beta) + a_1(\beta)\cdot\gamma_0 + a_2(\beta)\cdot{\gamma_0}^2 \tag{21}$$

are given in Table 2 together with relevant determination coefficients R^2.

Table 2. Approximation of the relation $y_0^*=y_0^*(\beta, \gamma_0)$.

β	a_0	a_1	a_2	R^2
0.00	0.6879	-0.4538	0.0646	1.0000
0.10	0.6881	-0.4542	0.0647	1.0000
0.20	0.6892	-0.4562	0.0652	1.0000
0.30	0.6922	-0.4622	0.0670	1.0000
0.40	0.6982	-0.4739	0.0706	1.0000
0.50	0.7087	-0.4953	0.0773	1.0000
0.55	0.7163	-0.5109	0.0823	1.0000
0.60	0.7260	-0.5312	0.0890	1.0000
0.65	0.7383	-0.5577	0.0981	1.0000
0.70	0.7540	-0.5926	0.1104	1.0000
0.75	0.7745	-0.6396	0.1277	1.0000
0.80	0.8017	-0.7050	0.1532	0.9999
0.85	0.8392	-0.8004	0.1928	0.9998
0.90	0.8949	-0.9514	0.2605	0.9993
0.95	0.9899	-1.2301	0.3979	0.9968

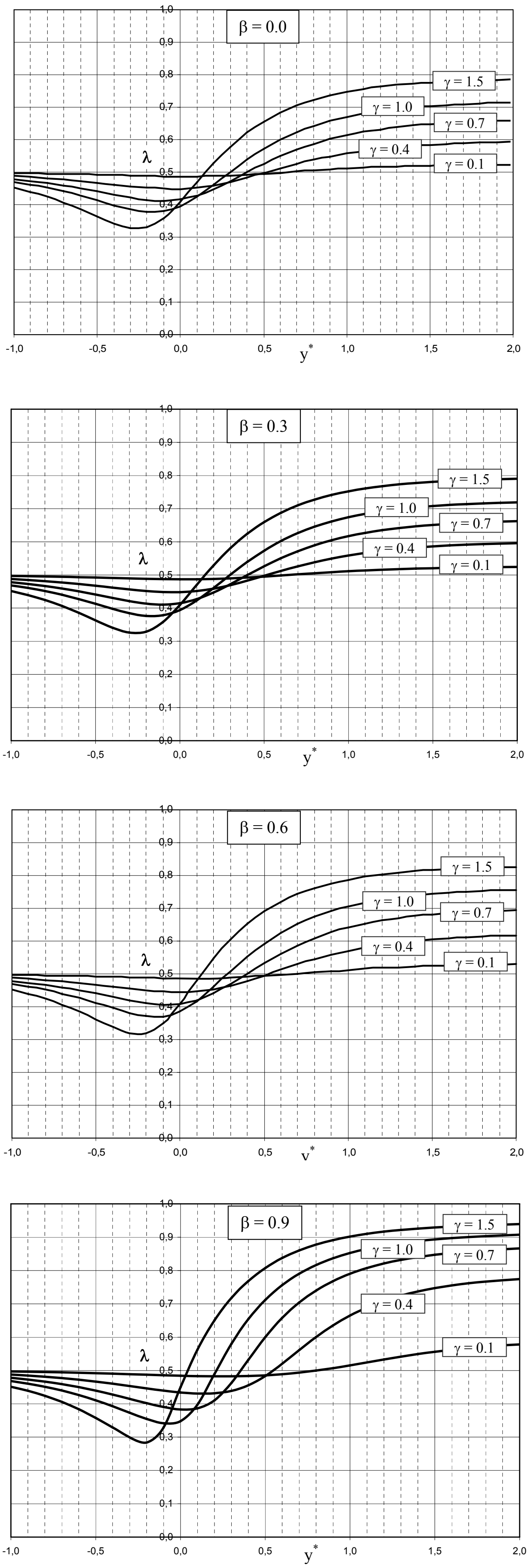

Fig. 3. Sample values of λ at different β, γ, and y^*.

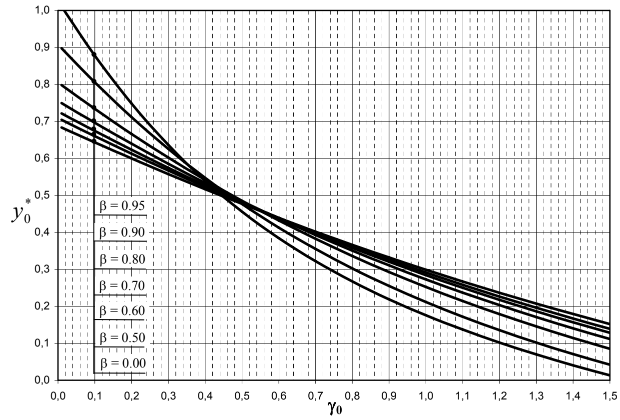

Fig. 4. Values y_0^* at different β and γ_0.

Application of linear interpolation to the trinomial approximation coefficients given in Table 2 for the first step of the energy equation integration in the main reaches of the example river network results in the following values:

– for the reach 2-1: β=0.513, y_0^*=0.269, γ_0=1.062, Δx_0=2,071 m (exact value – 2,076 m),
– for the reach 3-1: β=0.705, y_0^*= 0.301, γ_0= 0.919, Δx_0= 1,838 m (exact value – 1,847 m).

Conclusions

The simulations performed have revealed that numerical integration of the steady, gradually varied flow in looped networks of infinitely wide rectangular open channels may lead to significant errors of water level slopes and be the source of errors of flow values at particular reaches of the network. It has been shown that numerical integration of this equation is exact only for at most one integration step with its length being a function of the flow parameters. Therefore, in order to obtain the unbiased solution, one may either apply specified inconstant step length or modify trapezoidal rule by application of variable weight coefficients according to Eq. (11).

One can anticipate that similar relations exist for a rectangular channel of finite width; however, weight coefficients as optimum integration steps require further research in this case.

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